From Gauging Nonrelativistic Translations to N-Body Dynamics

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Abstract

We consider the gauging of space translations with time–dependent gauge functions. Using fixed time gauge of relativistic theory, we consider the gauge–invariant model describing the motion of nonrelativistic particles. When we use gauge–invariant nonrelativistic velocities as independent variables the translation gauge fields enter the equations through a $d \times (d+1)$ matrix of vielbein fields and their Abelian field strengths, which can be identified with the torsion tensors of teleparallel formulation of relativity theory. We consider the planar case ($d = 2$) in some detail, with the assumption that the action for the dreibein fields is given by the translational Chern–Simons term. We fix the asymptotic transformations in such a way that the space part of the metric becomes asymptotically Euclidean. The residual symmetries are (local in

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time) translations and rigid rotations. We describe the effective interaction of the $d = 2$ N-particle problem and discuss its classical solution for $N = 2$. The phase space Hamiltonian H describing two-body interactions satisfies a nonlinear equation $H = \mathcal{H}(\vec{x}, \vec{p}; H)$ which implies, after quantization, a non-standard form of the Schrödinger equation with energy dependent fractional angular momentum eigenvalues. Quantum solutions of the two-body problem are discussed. The bound states with discrete energy levels correspond to a confined classical motion (for the planar distance between two particles $r \le r_0$) and the scattering states with continuum energy correspond to the classical motion for $r > r_0$. We extend our considerations by introducing an external constant magnetic field and, for $N = 2$, provide the classical and quantum solutions in the confined and unconfined regimes.

1 Introduction

Our aim here is to discuss theories invariant under local time–dependent nonrelativistic translations ($\vec{x} = (x_1 \dots x_d)$):

$$x'_i = x'_i(\vec{x}, t) \tag{1.1}$$

supplemented by global space rotations ($x'_i = R_i{}^j x_j$) and global time translations

$$t' = t + a. \tag{1.2}$$

The usual approach to local coordinate invariance in nonrelativistic theory is to consider the limit $c \to \infty$ of a relativistic generally covariant theory. In particular it should be pointed out that the nonrelativistic limit of the Einstein action coupled to a relativistic point particle had been explicitly performed by Lusanna et al [1]. In this paper, following the earlier treatments by one of the present authors in the case of one–dimensional model [2,3], we shall impose nonrelativistic framework directly by constructing actions covariant under (1.1–2).

We shall consider the problem here by using the vielbein formulation of $(d+1)$–dimensional relativistic gravity, with d vectors $E^{\underline{a}}_{\mu}$ ($\underline{a} = 1, \dots, d; \mu = 0, 1, \dots d$) describing translational gauge fields[1] obtained from a relativistic $(d+1) \times (d+1)$ vielbein $E^{\underline{\rho}}_{\mu}$ ($\rho = 0, 1, \dots d$) by fixing completely $E^{\underline{0}}_{\mu}$. We will find that in the equation of motion for gauge–invariant nonrelativistic velocities

$$\xi^{\underline{a}} = E^{\underline{a}}_{\ i}\, \dot{x}_i + E^{\underline{a}}_{0}\,, \tag{1.3}$$

the derivative of the vielbein field will enter only through its Abelian field strength

$$T_{[\mu\nu]}^{\ \ \underline{a}} = \partial_\mu E^{\underline{a}}_{\nu} - \partial_\nu E^{\underline{a}}_{\mu} \tag{1.4}$$

[1]This idea goes back to the papers by Cho [4], Hayashi and Shirafuji [5,6]; for a review see [7]. Such framework leads to the so–called teleparallel formulation of relativity (see e.g. [5–10]), with vanishing curvature and nonvanishing torsion.

thus leading to the interpretation in terms of components of the torsion tensor.

The aim of this paper is to consider the dynamical consequences of the coupling of nonrelativistic particles in two space dimensions to nonstandard $D = 2+1$ gravity action. It has been found in analogous one-dimensional model [2,3] that gauging of nonrelativistic translations with Maxwell-like field action quadratic in the torsion tensor leads to confinement via the geometric bag formation. Further one can postulate[2] that such a confinement can occur on a new level of microscopic theories of fundamental interactions and does not appear in macroscopic "physical" gravity. The main result of the present paper is to show[3] that the geometric bag solutions which may describe confinement occur also in the planar case.

Our paper is organized as follows. Firstly, in Sect. 2 we shall consider d-dimensional nonrelativistic point particles, by using the Lagrangian framework covariant under (1.1). After considering in more detail the reparametrization–invariant nonrelativistic particle dynamics for $d = 1, 2$ and 3, in Sect. 3 we shall consider the field actions for vielbein fields, using known results from the teleparallel formulation of relativistic gravity in $1+1$, $2+1$ and $3+1$ dimensions. We shall find that the $d = 1$ action proposed in [2] is obtained from the well–known action in $D = 1+1$ Einstein–Cartan gravity [14]; for $D = 2+1$ we will consider the field action given by the so–called translational Chern–Simons term [15-17].

In Sect. 4 we solve the field equations having chosen the appropriate boundary conditions (gauge fixing). The general considerations of the N-particle classical particle dynamics in Sect. 5 is specialized in Sect. 6 to the two body problem. Its explicit classical solution is presented there. Quantization of the two body problem described by nonstandard Schrödinger equation and its solutions, in particular numerical calculations of energy levels and wave functions in the confinement regime are given in Sect. 7. The classical dynamics and quantized solutions in the presence of an external constant magnetic field are described in Sect. 8. Sect. 9 reports some of our conclusions and describes an outlook for further investigations.

2 Nonrelativistic Particles and their Covariant Coupling to Vielbein Fields

Let us assume that d space coordinates $\vec{x}(t) = (x_1(t) \ldots x_d(t))$ describe a trajectory of a d–dimensional point particle with dynamics invariant under the transformations (1.1). The velocities $\dot{x}_l \equiv \frac{dx_i}{dt}$ then transform under (1.1) as follows:

$$\dot{x}'_l = \frac{\partial x'_i}{\partial x_j}\,\dot{x}'_j + \frac{\partial x'_i}{\partial t}. \tag{2.1}$$

[2]It has been hypothesized already by Hehl that "torsion should arise in the microphysical realm" [11]. Some time ago [12] it was also shown that the gauge theory of gravity with torsion can lead to a confining potential.

[3]The preliminary results were presented in [13].

The formula (2.1) can be obtained from the $(d+1)$–dimensional "relativistic" formula $(x_\mu = (\vec{x}, x_0))$

$$\dot{x}'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,\dot{x}^\nu \tag{2.2}$$

if we assume that $x^0 = -x_0 = -t$, i.e.[4]

$$\dot{x}_0 = 1. \tag{2.3}$$

The preservation of (2.3) in any coordinate frame implies that

$$\frac{\partial x'_0}{\partial x_i} = 0 \qquad \frac{\partial x'_0}{\partial t} = 1 \Rightarrow x'_0 = x_0 + a \tag{2.4}$$

in accordance with (1.1).

It is well–known how to introduce the compensating gauge fields for the transformations (2.2); namely, we should replace the velocities by the world scalars (i.e. scalars under the transformations (2.2)) which are vectors in tangent space

$$\dot{x}^\mu \to \xi^{\underline{\mu}} = E^{\underline{\mu}}_{\ \nu}\dot{x}^\nu = \left(\xi^{\underline{0}}, \xi^{\underline{a}}\right). \tag{2.5}$$

Here the $(d+1)\times(d+1)$–bein $E_\nu{}^{\underline{\mu}}$ transforms as a covariant vector under the local transformations (1.1)

$$E'^{\underline{\mu}}_{\nu} = \frac{\partial x^\rho}{\partial x'^{\,\nu}}\,E^{\underline{\mu}}_{\rho} \tag{2.6}$$

and as a global $(d+1)$–dimensional vector under the Lorentz rotations in the tangent space. The imposition of relations (2.3–4) and their validity in any coordinate frame imply that one should choose

$$E^{\underline{0}}_{\ 0} = 1 \qquad E^{\underline{0}}_{\ i} = 0 \tag{2.7}$$

as then one gets $\xi^{\underline{0}} = \dot{x}^0 = -1$. The choice (2.7) we shall call the nonrelativistic gauge because it splits the $(d+1)$–dimensional Lorentz vector into a d–dimensional nonrelativistic vector and a scalar as well as because it implies the Newtonian notion of absolute time. One can write $(a, b = 1, \ldots d)$

$$E^{\underline{\mu}}_{\ \nu} = \begin{pmatrix} 1, & 0, & \ldots & 0 \\ e^{\underline{1}} & & & \\ \vdots & & h^{\underline{a}}_{\ i} & \\ e^{\underline{d}} & & & \end{pmatrix} \tag{2.8}$$

[4] The condition (2.3) can be treated as fixing the general reparametrization $x'_0 = x'_0(\vec{x}, x_0)$ of $(d+1)$-th coordinate.

and introduce the inverse vielbein $E^{\nu}{}_{\underline{\mu}}$ as follows:

$$E_{\underline{\mu}}{}^{\nu} = \begin{pmatrix} 1, & 0, & \dots & 0 \\ e^1 & & & \\ \vdots & & h^i{}_{\underline{a}} & \\ e^d & & & \end{pmatrix}, \tag{2.9}$$

where $h^{\underline{a}}{}_i h^i{}_{\underline{b}} = \delta^{\underline{a}}{}_{\underline{b}}$ and e^i, $e^{\underline{a}}$ are independent.

We replace the "flat" free nonrelativistic Lagrangian by [5]

$$\mathcal{L}_0 = \frac{m}{2}\dot{\vec{x}}^{\,2} \Rightarrow \mathcal{L}_0 + \mathcal{L}_{mt} = \frac{m}{2}\xi_{\underline{a}}\xi^{\underline{a}}. \tag{2.10}$$

Using formula (2.5) for the nonrelativistic covariantized velocity one obtains the following Euler–Lagrange equations of motion

$$h^{\underline{a}}{}_i\,\dot{\xi}_{\underline{a}} - T^{\underline{a}}_{i0}\,\xi_{\underline{a}} = T^{\underline{a}}_{ij}\,\xi_{\underline{a}}\,\dot{x}j \tag{2.11}$$

or, using (1.3) and then $E^i{}_{\underline{\mu}}E^{\mu}{}_0 = 0$, one gets $\dot{x}^j = h^j_{\underline{b}}\xi^{\underline{b}} - e^j$ and

$$\dot{\xi}_{\underline{c}} - h^i{}_{\underline{c}}\,h^j{}_{\underline{b}}\,T^{\underline{a}}_{ij}\,\xi_{\underline{a}}\,\xi^{\underline{b}} - h^i{}_{\underline{c}}\,T^{\underline{a}}_{i0}\,\xi_a = 0, \tag{2.12}$$

where the tensors $T^{\underline{a}}_{ij}$, $T^{\underline{a}}_{i0}$ are given by the formulae (see also (1.4))

$$T^{a}_{ij} = h^{\underline{a}}_{i,j} - h^{\underline{a}}_{j,i} \qquad T^{\underline{a}}_{i0} = h^{\underline{a}}_{i,0} - e^{a}_{\ ,i}\,. \tag{2.13}$$

In particular, for $d = 1$ we have only two zweibein–components $E^{\underline{1}}_0 = e$, $E^{\underline{1}}_1 = h$ and $T^{\underline{1}}_{ij} = 0$, $T^{\underline{1}}_{i0} = \partial_t h - \partial_x e$ (see [2]). In $d = 2$ we have

$$h^{\underline{a}}_i \;=\; \begin{pmatrix} h^{\underline{1}}_1, & h^{\underline{1}}_2 \\ h^{\underline{2}}_1, & h^{\underline{2}}_2 \end{pmatrix} \qquad\qquad e^{\underline{a}} = (e^{\underline{1}}, e^{\underline{2}}) \tag{2.14}$$

and it is easy to see that the term

$$T^{\underline{a}}_{ij}\,\xi_{\underline{a}}\,\dot{x}^j = \varepsilon_{ij}T^a\,\xi_{\underline{a}}\,\dot{x}^j \qquad (T^a = \frac{1}{2}\varepsilon^{ij}T^a_{ij})$$

does not vanish.

One can also write Lagrangian (2.10) for any d as

$$\mathcal{L}^{\rm cov}_0 = \frac{m}{2}\left(g_{ik}\,\dot{x}^i\,\dot{x}^k + 2g_{i0}\,\dot{x}^i + g_{00}\right), \tag{2.15}$$

[5]The d–dimensional indices, due to the Euclidean nonrelativistic metric, can be taken equivalently as lower or upper indices.

where

$$g_{ik} = h^{\underline{a}}_{i}\, h^{\underline{a}}_{k} \qquad g_{i0} = e^{\underline{a}}\, h^{\underline{a}}_{i} \qquad g_{00} = e^{\underline{a}}\, e^{\underline{a}} - 1. \tag{2.16}$$

We see that one can treat the g_{i0} components of the metric as describing the magnetic field coupled to the velocity of the particle[6] [18,19], and that the component g_{00} describes a potential. The Euler–Lagrange equations of motion now take the form[7]

$$\ddot{x}^i + \Gamma^i_{jk}\, \dot{x}^j\, \dot{x}^k = -g^{ij}\, \frac{\partial g_j{}^l}{\partial t}\, \dot{x}^l + g^{ij}\, F_{jl}\, \dot{x}^l + F^i\left[g_{ij}, g_{0i}, g_{00}\right], \tag{2.17}$$

where $\Gamma_{jk}^{\ l}$ is the symmetric Levi–Civita connection

$$\Gamma_{jk}^{\ l} = \frac{1}{2}\left(\frac{\partial g_{lj}}{\partial x_k} + \frac{\partial g_{lk}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_i}\right), \tag{2.18}$$

the field F_{jl} plays the role of the magnetic field strength ($F_{jl} = \partial_j A_l - \partial_l A_j$, with $A_j = g_{j0}$) and the force F^i is given by the formula

$$F^i = g^{ij}\, E_j, \tag{2.19}$$

where E_j plays the role of electric field strength

$$E_j = \frac{\partial A_0}{\partial x_j} - \frac{\partial A_j}{\partial t}, \qquad A_0 = g_{00}. \tag{2.20}$$

The most convenient form of the equations of motion depends on the type of field action for the vielbein field. The formulation with basic variables $\xi_{\underline{a}}$ is the best suited for the discussion of the interactions with a gravitational field described by an action functional which depends only on vielbeins and on the torsion variables (1.4).

3 Torsion Lagrangians and Coupled Particle–Torsion Field Systems

Following Möller [20] one can postulate vielbeins as fundamental variables in gravity theory and treat the metric as a derived quantity. Let us recall here that the covariantization of the Dirac equation requires vielbeins. Moreover, tetrads appear naturally in the geometric framework as translational gauge fields, providing the so–called teleparallel formulation of relativity theory [5–10].

Because eq. (2.11–12) involve only tetrads and torsion tensors, it is convenient to consider the free gravity actions which depend only on these field variables. In

[6]It is interesting to relate this observation to the Kaluza–Klein description of the electric charge in terms of the additional fifth dimension.

[7]The equations (2.16) appeared also in [1], as the nonrelativistic limit of a covariant relativistic particle model.

order to obtain nonrelativistic actions we can consider the relativistic $D = d + 1$–dimensional torsion actions and set the time component $T^0_{\mu\nu} = 0$, consistently with our nonrelativistic gauge condition (2.7). Because the explicit forms of the torsion actions depend strongly on dimensions, we shall consider below, separately, the three cases of $d = 1, 2$ and 3.

i) $d = 1$

It is known that in $1 + 1$ dimension the Einstein–Hilbert action is a topological invariant and so we can consider the action quadratic in curvature (see e.g. [21]) or in torsion [14]. The quadratic torsion action has the form

$$\overset{d=1}{\underset{rT}{S}} = \frac{1}{2\lambda} \int dt\, dx \cdot \det E \cdot T^{\underline{a}}_{\mu\nu}\, T^{\mu\nu}_{\underline{a}} \qquad \begin{cases} a = 0, 1 \\ \mu, \nu = 0, 1 \end{cases} \tag{3.1}$$

and reduces to the nonrelativistic field action considered in [2,3] when we observe that in the nonrelativistic gauge (2.7)

$$\det E = h\,. \tag{3.2}$$

Putting $T^{\underline{0}}_{\mu\nu} = 0$ and writing $F \equiv \frac{1}{h} T^{\underline{1}}_{\mu\nu}$ we get[8]

$$\overset{d=1}{\underset{nrT}{S}} = \frac{1}{2\lambda} \int dt\, dx\, h \cdot F^2. \tag{3.3}$$

ii) $d = 2$

The (2+1)–dimensional gravity with Hilbert–Einstein action is dynamically trivial as outside nonvanishing matter sources the space–time is flat (see e.g. [22]). This fact was one of the reasons why the alternative 3–dimensional topological gravity models – without torsion [23] and with torsion [16,17] had been considered. In particular, one can introduce as a candidate for a $D = 2 + 1$ gravity action the translational Chern–Simons term [16,17] ($t = x_0; \mu, \nu, \rho = 0, 1, 2, \underline{\alpha}, \underline{\beta} = 0, 1, 2; \eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1)$)

$$\overset{d=2}{\underset{nrT}{S}} = \frac{1}{\lambda} \int d^3x\, \varepsilon^{\mu\nu\rho}\, E^{\underline{\alpha}}_{\mu}\, T^{\underline{\beta}}_{\nu\rho}\, \eta_{\alpha\beta}. \tag{3.4}$$

If we pass to the nonrelativistic gauge (2.7) we obtain ($\underline{a}, \underline{b} = 1, 2$)

$$\overset{d=2}{\underset{nrT}{S}} = \frac{1}{\lambda} \int dt\, d^2x\, \varepsilon^{\mu\nu\rho}\, E^{\underline{a}}_{\mu}\, T^{\underline{a}}_{\nu\rho}. \tag{3.5}$$

The action (3.4) will be used in the next Section to derive and study various properties of the planar N–body interactions.

As later on we will study in detail the $d = 2$ case let us describe in some detail the planar coupled system of equations describing an interacting particle – torsion field system, with torsion fields described by the Abelian Chern–Simons action (3.5).

[8] See formula (18) in [2].

Introducing N trajectories $\vec{x}_\alpha(t) = x^i_\alpha(t)$ $(i = 1, 2; \alpha = 1, \ldots N)$ for N particles and the notation

$$E^{\underline{a}}_{\mu;\alpha}(t) \equiv E^{\underline{a}}_{\mu}(\vec{x}_\alpha(t), t), \tag{3.6}$$

the free action for N particles in $d = 2$ dimensions, in the first order formalism, can be written as

$$S^{(N)}_{\text{part}} = -\sum_{\alpha=1}^{N} m_\alpha \int dt \left[\frac{1}{2} \xi^{\underline{a}}_\alpha \xi^{\underline{a}}_\alpha - \xi^{\underline{a}}_\alpha \left(h^{\underline{a}}_{j;\alpha} \dot{x}^j_\alpha + e^{\underline{a}}_\alpha \right) \right], \tag{3.7}$$

thus providing the constraint formula

$$\xi^{\underline{a}}_\alpha = h^{\underline{a}}_{j;\alpha} \dot{x}^j + e^{\underline{a}}_\alpha. \tag{3.8}$$

The action (3.7) can be written in the field form, using the mass density function, as

$$S^{(N)}_{\text{part}} = -\int dt d^2x \left[\frac{1}{2} \xi^{\underline{a}} \xi^{\underline{a}} - \xi^{\underline{a}} \left(h^{\underline{a}}_j \dot{x}^j + e^{\underline{a}} \right) \right] \rho_r(\vec{x}; \vec{x}_1(t) \ldots \vec{x}_N(t)), \tag{3.9}$$

where

$$\rho_N(\vec{x}; \vec{x}_1(t) \ldots \vec{x}_N(t)) = \sum_{\alpha=1}^{N} m_\alpha \, \delta^{(2)} \left(\vec{x} - \vec{x}_\alpha(t) \right). \tag{3.10}$$

The field action (3.5) thus takes the form of the field action $(i, j, k = 1, 2)$

$$S_{\text{field}} = \frac{1}{\lambda} \int dt d^2x \left(e^{\underline{a}} B^{\underline{a}} - \varepsilon_{jk} \, h^{\underline{q}}_j \mathcal{E}^{\underline{a}}_k \right), \tag{3.11}$$

where

$$B^{\underline{a}} = \varepsilon_{jk} \, \partial_j \, h^{\underline{a}}_k \qquad \mathcal{E}^{\underline{a}}_k = \partial_t \, h^{\underline{a}}_k - \partial_k \, e^{\underline{a}}. \tag{3.12}$$

The fields $B^{\underline{a}}$ and $\mathcal{E}^{\underline{a}}_k$ plays the role of the magnetic and electric fields, respectively, with the internal $O(2)$ index $\underline{a}$ describing the $d = 2$ nonrelativistic rotation group[9].

We can now derive the coupled equations describing the $d = 2$ particle–field system, described by the action $S = S^{(N)}_{\text{part}} + S_{\text{field}}$ (see (3.7) and (3.9)). The equations for the particle trajectories (see (2.11)), having used the notation (3.12), then take the form:

$$h^{\underline{a}}{}_i \, \dot{\xi}^{\underline{a}}_\alpha - \varepsilon_{ij} \, B^{\underline{a}}_\alpha \, \xi^{\underline{a}}_\alpha \, \dot{x}^j - \mathcal{E}^{\underline{a}}_{j;\alpha} \, \xi^{\underline{a}}_\alpha = 0. \tag{3.13}$$

We can now pass to the form (2.11) and use the following explicit form for the nonrelativistic dreibein (2.9)

$$E^{\nu}{}_{\mu} = \frac{1}{d} \begin{pmatrix} d & 0 & 0 \\ e^{\underline{2}} h^{\underline{1}}{}_2 - e^{\underline{1}} h^{\underline{2}}{}_2, & h^{\underline{2}}{}_2, & -h^{\underline{1}}{}_2, \\ e^{\underline{1}} h^{\underline{1}}{}_1 - e^{\underline{2}} h^{\underline{1}}{}_1, & -h^{\underline{2}}{}_1, & h^{\underline{1}}{}_1 \end{pmatrix}, \tag{3.14}$$

[9] In the vielbein formalism the rotation group (Lorentz in the relativistic case, Euclidean in the nonrelativistic gauge) plays the role of an internal symmetry.

where $d = \det(h^{l}{}_{j}) = h^{1}{}_{1}h^{2}{}_{2} - h^{1}{}_{2}h^{2}{}_{1}$.

The field equations for the dreibein fields then take the form:

$$\mathcal{E}^{\underline{a}}_{k}(\vec{x},t) = -\frac{\lambda}{2}\sum_{\alpha} m_{\alpha}\varepsilon_{kj}\,\dot{x}^{j}_{\alpha}\cdot\xi^{\underline{a}}_{\alpha}\cdot\delta^{(2)}(\vec{x}-\vec{x}_{\alpha}(t)), \tag{3.15a}$$

$$B^{\underline{a}}(\vec{x},t) = -\frac{\lambda}{2}\sum_{\alpha} m_{\alpha}\xi^{\underline{a}}_{\alpha}\delta^{(2)}(\vec{x}-\vec{x}_{\alpha}(t)). \tag{3.15b}$$

We see, therefore, that the dreibein field equations, consistently with the general property of the $d = 2+1$ gravity, imply that the space–time is flat outside of the matter sources.

In the next section we shall try to solve the system of equations (3.13) and (3.15a-b) by using the technique of singular gauge transformations (see e.g. [24–28]).

iii) $d = 3$

In $3+1$ dimensions we can write three independent quadratic torsion actions [5–10]. It is interesting to observe that there are special linear combinations of quadratic torsion terms which provide four–dimensional Hilbert–Einstein action. The detailed consideration of coupled nonrelativistic particle – torsion fields systems[10] is postponed to a future investigation.

4 Solution of the Field Equations

4.1 Gauge fixing and residual symmetry

The set of equations (3.15a-b) can be rewritten as (see (3.5)):

$$\epsilon^{\mu\nu\rho}\partial_{\mu}E^{\underline{a}}_{\nu} = -\frac{\lambda}{2}\sum_{\alpha}\xi^{\underline{a}}_{\alpha}\,\dot{x}^{\rho,\alpha}\,\delta(\underline{x}-x^{\alpha}), \tag{4.1}$$

where (3.15a) and (3.15b) are obtained respectively by putting $\mu = 1,2$ ($\mu \equiv \alpha$) and $\mu = 0$.

The general solution of (4.1) can be written in the pure gauge form

$$E^{\underline{a}}_{\mu}(\vec{x},t) = \widetilde{E}\,{}^{\underline{a}}_{\mu}(\vec{x},t) + \partial_{\mu}\Lambda^{\underline{a}}(\vec{x},t) = \partial_{\mu}\tilde{\Lambda}\,{}^{\underline{a}}(\vec{x},t) \tag{4.2a}$$

where

$$\widetilde{E}\,{}^{\underline{a}}_{\mu}(\vec{x},t) = -\frac{\lambda}{4\pi}\partial_{\mu}\sum_{\alpha}\xi^{\underline{a}}_{\alpha}\,\Phi(\vec{x}-\vec{x}_{\alpha}) \tag{4.2b}$$

and

- $\Lambda^{\underline{a}}$ is an $O(2)$–vector valued pair of regular gauge functions,

[10] In our framework, due to the relation (1.4), the local frame fields E^{a}_{μ} can be called torsion potentials.

- $\Phi(\vec{x})$ is a singular gauge function satisfying the following equation (see e.g. [26–28])

$$\epsilon^{ij}\partial_i\partial_j\Phi(\vec{x}) = 2\pi\,\delta(\vec{x}\,) \tag{4.3}$$

thus expressing the singular nature of the first term in (4.2b). As a solution of (4.3) we can take

$$\Phi(\vec{x}\,) = \arctan\frac{x_2}{x_1} \tag{4.4}$$

i.e.

$$\partial_k\Phi(\vec{x}) = -\varepsilon_{kl}\partial_l \ln|\vec{x}\,|. \tag{4.5}$$

The function $\partial_k\Phi(\vec{x})$ can be regularized in such a way that it has a well defined limit for $\vec{x}\to 0$, e.g. we can replace $\ln|\vec{x}\,|$ in (4.5) by [28]

$$\ln|\vec{x}\,| \longrightarrow \ln^{(\epsilon)}(\vec{x}) := \frac{1}{\epsilon\pi}\int d^2y\,\ln|\vec{x}-\vec{y}\,|e^{-\frac{y^2}{\epsilon}}. \tag{4.6}$$

In this case we find

$$\begin{aligned} &i) && \ln^{(\epsilon)}(\vec{x}) \xrightarrow{\epsilon\searrow 0} \ln|\vec{x}|, \\ &ii) && \lim_{\vec{x}\to 0}\varepsilon_{kl}\partial_l\ln^{(\epsilon)}(\vec{x}) \longrightarrow 0 \qquad \forall\varepsilon>0. \end{aligned} \tag{4.7}$$

Let us note that the solutions for the fields $E_\mu^{\underline{a}}(\vec{x},t)$ with asymptotically nonvanishing gauge function $\Lambda^{\underline{a}}$ do not solve the Hamilton's variational principle for the field action (3.11). The bad asymptotic behaviour for $r\to\infty$ of dreibeins $E_\mu^{\underline{a}}(\vec{x},t)$ leads to the appearence of nonvanishing surface integrals and, in consequence, our $E_\mu^{\underline{a}}(\vec{x},t)$ do not minimize the action. Such a situation arises also in General Relativity (see e.g. [29]). In the following we will argue along similar lines by choosing an appropriate asymptotic form for $E_\mu^{\underline{a}}$ and adding two surface integrals to the action.

To do this we decompose

$$E_\mu^{\underline{a}}(\vec{x},t) \;=\tilde{E}_\mu^{\underline{a}}(\vec{x},t) \;+\; E_\mu^{\mathrm{as}\,\underline{a}} \tag{4.8}$$

where we require that the new field variables $\tilde{E}_\mu^{\underline{a}}(\vec{x},t)$ satisfy

$$\tilde{E}_\mu^{\underline{a}}(\vec{x},t) \;\to\; O(r^{-1}) \quad \text{as} \quad r\to\infty \tag{4.9}$$

and we assume that the asymptotic parts $E_\mu^{as\underline{a}}$ are given as functions of t only. Then from the pure gauge form

$$E_\mu^{as\underline{a}}(t) = \partial_\mu\Lambda^{\underline{a}}(\vec{x},t). \tag{4.10}$$

we obtain

$$\Lambda^i(\vec{x},t) = x^i - a^i(t) \tag{4.11}$$

giving us

$$E_0^{as\underline{a}} \;=\; -\dot{a}^{\underline{a}}(t) \;=:\; -v^{\underline{a}}(t), \qquad E_i^{as\underline{a}} \;=\; \delta_i^{\underline{a}}, \tag{4.12}$$

where we have chosen the factor in front of x^i to be equal to one in order to have, asymptotically, the Euclidean metric.

The choice (4.11) for Λ^i breaks asymptotically the invariance with respect to local space translations (1.1). In the formula (4.2a) only the fields $\tilde{E}^{\underline{a}}_{\mu}$ transform covariantly with respect to those local translations which preserve the asymptotic behaviour (4.9). However, as under general coordinate transformations the functions $\widetilde{\Lambda}^{\,i}$ are scalars, the changes $\delta\,\widetilde{\Lambda}^{\,i}$ under (1.1) of both the singular and regular parts of $\widetilde{\Lambda}^{\,i}$ must separately vanish and we obtain

$$\delta x^i \;=\; \delta a^i(t), \tag{4.13}$$

where x^i and a^i transform as vectors under rotatons in tangent space. Therefore, as a residual symmetry, we obtain translations, local in time, and rigid rotations (see also [30-32] for a similar situation in General Relativity).

Putting (4.8) and (4.11) into the Chern–Simons action (see (3.5) and (3.11)) we find (modulo a total time derivative)

$$L_{field} \;=\; \tilde{L}_{field} \;-\; I_1 \;-\; I_2 \tag{4.14}$$

where

$$\tilde{L}_{field} := \; L_{field}[\tilde{E}^{\underline{a}}_{\mu}] \tag{4.15}$$

and the integrals $I_{1,2}$ are defined by

$$I_1 \;=\; \tfrac{1}{\lambda} \int d^2x\, v^{\underline{a}} \cdot \epsilon^{ij}\, \partial_i\, \tilde{E}^{\underline{a}}_{j}(\vec{x},t) \tag{4.16}$$

$$I_2 \;=\; \tfrac{1}{\lambda} \int d^2x\, \epsilon^{ij}\, \partial_i\, \tilde{E}^{\underline{j}}_{0}. \tag{4.17}$$

Next we use the Stokes' theorem to rewrite $I_{1,2}$ as

$$I_1 \;=\; \tfrac{1}{\lambda} \lim_{r\to\infty} r v^{\underline{a}} \int_0^{2\pi} d\varphi\, \Big(-\tilde{E}^{\underline{a}}_1\, sin(\varphi) \;+\; \tilde{E}^{\underline{a}}_2\, cos(\varphi)\Big)\,, \tag{4.18}$$

$$I_2 \;=\; \tfrac{1}{\lambda} \lim_{r\to\infty} r \int_0^{2\pi} d\varphi\, \Big(-\tilde{E}^{1}_0\, sin(\varphi) \;+\; \tilde{E}^{2}_0\, cos(\varphi)\Big)\,. \tag{4.19}$$

Given the asymptotic behaviour (4.9) the boundary integrals $I_{1,2}$ exist but they do not vanish, in general, and so only the modified field Lagrangian

$$\tilde{L}_{field} \;=\; L_{field} \;+\; I_1 \;+\; I_2 \tag{4.20}$$

has well defined functional derivatives with respect to the field $\tilde{E}^{\underline{a}}_{\mu}$.

Due to the asymptotic behaviour (4.9) the variations of $\tilde{E}^{\underline{a}}_{\mu}$ and $v^{\underline{a}}$ are independent of each other *i.e.* the fields $\tilde{E}^{\underline{a}}_{\mu}$ and $v^{\underline{a}}$ appear as new variables. The property that in a $(2+1)$-dimensional gravity boundary terms give rise to additional degrees of freedom has been shown also in [32].

The particle action now takes the form

$$S^{(N)}_{\mathrm{part}}[x,\xi,E] \;=\; S^{(N)}_{\mathrm{part}}[x,\xi,\tilde{E}] \;+\; \int dt\, \left[\sum_\alpha \xi^{\underline{a}}_\alpha \cdot \dot{x}^{\underline{a}}_\alpha \;-\; v^{\underline{a}} \cdot \sum_\alpha \xi^{\underline{a}}_\alpha \right]. \tag{4.21a}$$

and the modified field action $\tilde{S}_{\mathrm{field}} = S_{\mathrm{field}}[\widetilde{E}]$ (see (4.20)) is given by $S_T^{(N\,R)}$ (eq. 3.5) but taken as a function of $\tilde{E}$

$$S_{\mathrm{field}}[\widetilde{E}] = S_T^{(N\,R)}[\widetilde{E}] \tag{4.21b}$$

We see that in the action (4.21) there are separated the variables describing the "bulk" ($\widetilde{E}\,^{\underline{a}}_{\mu}$) and asymptotic behaviour (v^a).

The new variables ($v^{\underline{a}}$) describing local gauge degrees of freedom appear as the Lagrange multipliers in (4.21a) and are not determined by the EOM (cp. [33]). By fixing it as a constant we obtain Galilei invariance as the residual symmetry.

If we vary $\tilde{E}^{\underline{a}}_\mu$ we obtain the field equations (4.1) whose solutions should be taken, in accordance with (4.9), as

$$\tilde{E}^{\underline{a}}_\mu(\vec{x},t) \;=\; -\frac{\lambda}{4\pi}\,\partial_\mu \sum_\alpha \xi^{\underline{a}}_\alpha(t)\Phi(\vec{x}-\vec{x}_\alpha(t)). \tag{4.22}$$

By varying S with respect to $v^{\underline{a}}$ we obtain the constraint

$$\sum_\alpha \xi^{\underline{a}}_\alpha \;=\; 0. \tag{4.23}$$

We note that only the presence of the constraint (4.23) yields the correct asymptotic behaviour as well as the correct rotational properties of $\tilde{E}^{\underline{a}}_0$ because

- from (4.22) we have

$$\tilde{E}^{\underline{a}}_0(\vec{x},t) \;\to\; -\frac{\lambda}{4\pi}\sum_\alpha \dot{\xi}^{\underline{a}}_\alpha\,\Phi(\vec{x}) \;+\; O(r^{-1})\,, \quad \text{as} \quad r\to\infty \tag{4.24}$$

  which agrees with (4.9) if (4.23) holds.

- for a rotation by an angle φ in the 1-2 plane we have

$$\Phi \to \Phi + \varphi \tag{4.25}$$

  and so with (4.24) we get

$$\tilde{E}^{\underline{a}}_0 \;\to\; \tilde{E}^{\underline{a}}_0 \;-\; \Phi\,\frac{\lambda}{4\pi}\sum_\alpha \dot{\xi}^{\underline{a}}_\alpha, \tag{4.26}$$

  which agrees with (2.6) only if (4.23) holds.

4.2 Conservation Laws

Due to the constraint (4.23) the total momentum of the system vanish. To see this let us note that after gauge fixing our Lagrangian L is invariant with respect to space translations local in time, and rigid rotations. According to Noether's theorem, the invariance of L with respect to $\delta\vec{x}$ leads to a conserved quantity $C[\delta\vec{x}]$ (see [34], eq. (6.29c) for the corresponding case of the Chern–Simons electrodynamics):

$$C[\delta\vec{x}] \;=\; \sum_\alpha p^{\underline{a}}_\alpha \cdot \delta x^{\underline{a}}_\alpha \;+\; \frac{2}{\lambda}\int \, d^2x \, B^{\underline{a}} \tilde{E}^{\underline{a}}_k \, \delta x_k \;=\; C_{part} \;+\; C_{field}. \tag{4.27}$$

Inserting into (4.27) the expression (3.15b) for $B^{\underline{a}}$ and (4.22) for $\tilde{E}_k{}^{\underline{a}}$ and taking into account the definition of the canonical particle momentum $p^{\underline{a}}_\alpha = \frac{\partial L}{\partial \dot{x}^{\underline{a}}_\alpha}$ we get

$$C[\delta\vec{x}] \;=\; \sum_\alpha \xi^{\underline{a}}_\alpha \, \delta x^{\underline{a}}_\alpha \tag{4.28}$$

Taking $\delta\vec{x}$ as describing respectively space translations ($\delta x^{\,\underline{a}}_{\,\alpha} \;=\; \delta a^{\,\underline{a}}$) and rotations ($\delta x^{\,\underline{a}}_{\,\alpha} = \varepsilon^{\,\underline{ab}} x^{\underline{b}}_\alpha \cdot \delta\alpha$) and denoting the corresponding conserved quantities by P_i and J, we obtain due to (3.15b) and (4.22) for P_i the formula

$$P^i \;= P^i_{\mathrm{part}} = \sum_\alpha \xi^i_\alpha \tag{4.29}$$

and for J we get

$$J \;=\; \sum_\alpha \, \epsilon_{ij} \, x_{i,\alpha} \, \xi_{j,\alpha}. \tag{4.30}$$

Observe that J is different from the total canonical particle angular momentum J_{part}. The field contribution J_{field}, which is obtained from (3.15b) (4.22) and (4.23), namely,

$$J_{field} \;=\; -\frac{\lambda}{8\pi} \, \sum_\alpha \xi^i_\alpha \, \xi^i_\alpha \tag{4.31}$$

is conserved separately because it is proportional to the N-particle Hamiltonian to be given in section 5.2.

The fact that $J \neq J_{part}$ will play an important role in the quantization of the system (see Sect. 7).

In section 3 we showed that the residual symmetry contain the translations local in time. The conserved quantities P^i, which vanish according to (4.23) and (4.29) in the physical phase space are the generators of this symmetry in an extended phase space. The same result has been obtained for $(2+1)$-dimensional gravity [35] and by one of the present authors (PSC) in the one–dimensional case [2], [3]. For the similar case of local (in time) rotations it has been shown that the corresponding rotation generator is a first class constraint which vanish in the physical part of the phase space (cp. [36], [37]).

5 Classical Particle Dynamics (N Body Problem)

5.1 General Properties of the Solutions

Let us look at the classical equations of motion and their solutions. The equations to solve are then

$$\xi^{\underline{a}}_{\alpha} \;=\; E^{\underline{a}}_{\alpha,i}\dot{x}^{i,\underline{a}} \;+\; E^{\underline{a}}_{\alpha,0} \tag{5.1}$$

and

$$\dot{\xi}^{\underline{a}}_{\alpha} \cdot E^{\underline{a}}_{\alpha,i} \;+\; \xi^{\underline{a}}_{\alpha} \cdot F^{\underline{a}}_{\mu i,\alpha}\dot{x}^{\mu,\alpha} \;=\; 0 \tag{5.2}$$

for the particle coordinates $\{\vec{x}^{\alpha}(t)\}$ after the insertion of the solution (4.2) for $E^{\underline{a}}_{\alpha,\mu}$ into these equations.

However, from (4.1) we obtain for $F^{\underline{a}}_{\mu\nu}$

$$F^{\underline{a}}_{\mu\nu} \;=\; -\frac{\lambda}{2}\,\epsilon_{\mu\nu\rho}\sum_{\beta}\xi^{\underline{a}}_{\beta}\,\dot{x}^{\rho,\beta}\,\delta(\vec{x}-\vec{x}^{\beta}). \tag{5.3}$$

Thus the second term in (5.2), due to the antisymmetry of the ϵ tensor, can be rewritten as

$$\xi^{\underline{a}}_{\alpha} \cdot F^{\underline{a}}_{\mu i,\alpha}\dot{x}^{\mu,\alpha} \;=\; -\frac{\lambda}{2}\epsilon_{\mu i\rho}\sum_{\beta\neq\alpha}\xi^{\underline{a}}_{\beta}\;\xi^{\underline{a}}_{\alpha}\,\dot{x}^{\rho,\beta}\dot{x}^{\mu,\alpha}\delta(\vec{x}_{\alpha}-\vec{x}_{\beta}) \tag{5.4}$$

which is infinite for coinciding particle positions and vanishes otherwise. Therefore our configuration space contains only noncoinciding particle positions.

From eq. (5.2) and (5.4) we have

$$\dot{\xi}^{\underline{a}}_{\alpha} \cdot E^{\underline{a}}_{\alpha,i} \;=\; 0, \tag{5.5}$$

which leads, for points in the configuration space where the metric is non-degenerate, to

$$\dot{\xi}^{\underline{a}}_{\alpha} \;=\; 0. \tag{5.6}$$

Using this result we obtain for $\xi^{\underline{a}}_{\alpha}$ given by (1.3) from (4.2) and (4.10–12)

$$\xi^{\underline{a}}_{\alpha} \;=\; \dot{x}^{\underline{a}}_{\alpha} - v^{\underline{a}} - \frac{\lambda}{4\pi}\sum_{\beta\neq\alpha}\xi^{\underline{a}}_{\beta}\left(\dot{x}^{\underline{b}}_{\alpha\beta}\cdot\partial_b\Phi(\vec{x}_{\;\alpha\beta})\right), \tag{5.7}$$

where we have defined

$$x^{\underline{b}}_{\alpha\beta} := x^{\underline{b}}_{\alpha} - x^{\underline{b}}_{\beta}. \tag{5.8}$$

For the particular case of a particle motion on a line we find, from the last two expressions, that

$$\ddot{x}^{\underline{a}}_{\alpha} \;=\; 0 \tag{5.9}$$

i.e. the free motion.

This is also easily seen from (see (1.3c) in [26])

$$\dot{x}^{a}_{\alpha\beta} \cdot \partial_a \Phi(x_{\alpha\beta}) \,=\, \vec{x}_{\,\alpha\beta} \wedge \frac{\dot{\vec{x}}_{\,\alpha\beta}}{|\vec{x}_{\alpha\beta}|^2} \,=\, 0 \tag{5.10}$$

as $\vec{x}_{\alpha\beta}$ and $\dot{\vec{x}}_{\alpha\beta}$ are parallel for a motion along a line.

We should say also a few words about the degenerate case *i.e.* when (5.6) is not necessarily satisfied.

Then we have for ξ^{a}_{α} from (4.2) and (4.10–12)

$$\xi^{a}_{\alpha} = \dot{x}^{a}_{\alpha} - v^{a} - \frac{\lambda}{4\pi} \sum_{\beta\neq\alpha} \xi^{a}_{\beta} \left(\dot{x}^{b}_{\alpha\beta} \cdot \partial_b \Phi(\vec{x}_{\alpha\beta}) \right) - \frac{\lambda}{4\pi} \sum_{\beta\neq\alpha} \dot{\xi}^{a}_{\beta} \left(\Phi(\vec{x}_{\alpha\beta}) - \Phi(0) \right), \tag{5.11}$$

where $\Phi(0)$ has to be defined by an appropriate regularization procedure. But for any such procedure the last term in the last formula breaks rotational invariance. Thus, although $\tilde{E}^{a}_{0}(\vec{x},t)$ is rotationally invariant this is not the case for

$$\tilde{E}^{a}_{\alpha,0} \,=\, \lim_{\vec{x}\to\vec{x}_\alpha} \tilde{E}\,^{a}_{0}(\vec{x},t) \tag{5.12}$$

if (5.6) does not hold.

Hence the degenerate case can be considered as unphysical and further will not be discussed.

5.2 Reduced Hamiltonian/Lagrangian for the Non–Degenerate Case

Applying the Legendre transformation to the total Lagrangian given by (4.21) we find the following Hamiltonian

$$\begin{aligned} H \;=\; & -\textstyle\int dx^2\, \tilde{E}\,^{a}_{0} \cdot \left(\frac{2}{\lambda}\epsilon^{ij}\partial_i \,\tilde{E}\,^{a}_{j} + + \sum\limits_{\alpha} \xi^{a}_{\alpha}\, \delta(\vec{x}-\vec{x}_\alpha) \right) \\ & + \tfrac{1}{2} \sum\limits_{\alpha} \xi^{a}_{\alpha}\xi^{a}_{\alpha} + v^{a} \cdot \sum\limits_{\alpha} \xi^{a}_{\alpha}. \end{aligned} \tag{5.13}$$

The gauge field $\tilde{E}\,^{a}_{0}$ is a Lagrange multiplier field, whose variation gives the constraint

$$\epsilon^{ij}\partial_i \,\tilde{E}\,^{a}_{j}(\vec{x},t) \,=\, -\frac{\lambda}{2} \sum_{\alpha} \xi^{a}_{\alpha}\, \delta(\vec{x}-\vec{x}_\alpha(t)) \tag{5.14}$$

which corresponds to the $\mu = 0$ term in (4.1).

When if we put this expression into the Hamiltonian we find

$$H^{(N)} \,=\, \frac{1}{2} \sum_{\alpha} \xi^{a}_{\alpha}\xi^{a}_{\alpha} + v^{a} \cdot \sum_{\alpha} \xi^{a}_{\alpha}. \tag{5.15}$$

This is a free Hamiltonian with a constraint and the dynamics is contained in the non-trivial symplectic structure. [11]

In order to complete the Legendre transformation we have to express the auxiliary variable $\xi^{\underline{a}}_\alpha$ in terms of the canonical variables $\{x^{\underline{a}}_\alpha,\, p^{\underline{a}}_\alpha\}$. This can be done in a unique way iff

$$det\left(\frac{\partial p^i_\alpha}{\partial \xi^j_\beta}\right) \neq 0, \tag{5.16}$$

where

$$\begin{aligned} p^i_\alpha := \tfrac{\partial L}{\partial \dot{x}^i_\alpha} \;&=\; \xi^{\underline{a}}_\alpha \cdot E^{\underline{a}}_{\alpha\, i} \;= \\ =\; \xi^i_\alpha \;-\; \tfrac{\lambda}{4\pi} &\sum_{\beta\neq\alpha} (\xi^{\underline{a}}_\alpha \cdot \xi^{\underline{a}}_\beta)\partial_i \Phi(\vec{x}_{\,\alpha\beta}). \end{aligned} \tag{5.17}$$

The condition (5.6) defining the non-degenerate metric now arises as a consequence of the EOM. When we apply an inverse Legendre transformation to (5.15) we get

$$L_{red} \;=\; \sum_\alpha p^{\underline{a}}_\alpha \cdot \vec{x}^{\,\underline{a}}_\alpha \;-\; H_{red}\,. \tag{5.18}$$

Varying S_{red} with respect to x^i_α we obtain using (5.17)

$$\sum_\beta \frac{\partial p^i_\alpha}{\partial \xi^{\underline{a}}_\beta}\, \dot{\xi}^{\underline{a}}_\beta \;=\; 0 \tag{5.19}$$

which leads only to the trivial solution

$$\dot{\xi}^{\underline{a}}_\alpha \;=\; 0, \tag{5.20}$$

if (5.16) holds. We conclude therefore that the existence of the Legendre transformation implies a non-degenerate metric.

Varying S_{red} with respect to $v^{\underline{a}}$ and $\xi^{\underline{a}}_\alpha$ we obtain, respectively, the constraints

$$\sum_\alpha \xi^{\underline{a}}_\alpha \;=\; 0, \tag{5.21}$$

in accordance with the relation (4.23) obtained in geometric way by suitable gauge fixing and

$$\xi^{\underline{a}}_\alpha \;=\; \dot{x}^{\underline{a}}_\alpha \;-\; v^{\underline{a}} \;-\; \frac{\lambda}{4\pi}\sum_{\beta\neq\alpha}(\xi^{\underline{a}}_\beta \cdot \left(\dot{x}^{\,b}_{\alpha\beta} \cdot \partial_b \Phi(\vec{x}_{\,\alpha\beta})\right)\,. \tag{5.22}$$

Alternatively, we may consider the Hamiltonian (5.15) in the $\{x^{\underline{a}}_\alpha,\, \xi^{\underline{a}}_\alpha\}$-space. In such a case the free Hamiltonian (5.15) will be endowed with the nontrivial symplectic structure.

[11] Compare with [34] where a similar discussion of the Chern–Simons electrodynamics for point particles is given.

5.3 Transformation to flat coordinates

Let us recall from Sect. 2 that the space parts of our dreibeins $E_i^{\underline{a}}$ determine the metric g_{ij} in the two-dimensional space

$$g_{ij}(\vec{x},t) := \,(E_i^{\underline{a}}(\vec{x},t)\cdot E_j^{\underline{a}}(\vec{x},t)) \tag{5.23}$$

with the line element given by

$$ds^2 \,=\, g_{ij}\,dx^i\,dx^j. \tag{5.24}$$

But according to (4.2) the $E_i^{\underline{a}}$ are singular due to the formula (we put $\tilde{\Lambda}^{\underline{a}} \equiv y^{\underline{a}}$)

$$E_i^{\underline{a}} \,=\, \partial_i\, y^{\underline{a}}(\vec{x},t) \tag{5.25}$$

with the singularities located at the particle positions $\{x_{\alpha}^{\underline{a}}(t)\}, (\alpha = 1,\ldots N)$. We see that our metric is locally flat[12] and we have

$$ds^2 \,=\, (dy^{\underline{a}}\cdot dy^{\underline{a}}) \tag{5.26}$$

in $R^2 - \{x_{\alpha}^{\underline{a}}(t)\}, (\alpha = 1,\ldots N)$.

We can now show that the variables $\{y_{\alpha}^{\underline{a}}(t)\}$ are canonically conjugate to the covariantized velocities $\{\xi_{\alpha}^{\underline{a}}(t)\}$. To show this we consider $F(\{\xi_{\alpha}^{\underline{a}}, x_{\alpha}^{\underline{a}}\})$ defined as follows:

$$F := \sum_{\alpha} \xi_{\alpha}^{\underline{a}}\cdot x_{\alpha}^{\underline{a}} \,-\, \frac{\lambda}{8\pi} \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} (\xi_{\beta}^{\underline{a}}\cdot\xi_{\alpha}^{\underline{a}})\,\phi(\vec{x}_{\,\alpha\beta}) \tag{5.27}$$

F is a generating function for the canonical transformation

$$\{x_{\alpha}^{\underline{a}}, p_{\alpha}^{\underline{a}}\} \quad\rightarrow\quad \{y_{\alpha}^{\underline{a}}, \xi_{\alpha}^{\underline{a}}\} \tag{5.28}$$

at the points in phase space for which

$$det\left(\frac{\partial^2 F}{\partial\xi_i \partial x_j}\right) \,\neq\, 0. \tag{5.29}$$

Indeed, the relation (5.17) takes the form

$$p_{\alpha,i} = \frac{\partial F}{\partial x_{\alpha,i}} \tag{5.30}$$

and from (5.25), (4.2) and (4.8–11) it follows

$$y_{\alpha,i} = \frac{\partial F}{\partial \xi_{\alpha,i}} \tag{5.31}$$

Thus we conclude that the motion is free in the part of the phase space $\{y_{\alpha}^{\underline{a}}, \xi_{\alpha}^{\underline{a}}\}_{\sum\limits_{\alpha}\xi_{\alpha}^{\underline{a}}=0}$ restricted by the condition (5.29) and with substracted coinciding particle positions.

[12] Compare with (2+1) dimensional gravity (cp. [22])

6 Two Body Problem

6.1 Symplectic Structure

We define:

$$\xi^{\underline{a}} := \frac{1}{2}\,(\xi_1^{\underline{a}} - \xi_2^{\underline{a}})\,, \qquad x^{\underline{a}} := x_1^{\underline{a}} - x_2^{\underline{a}}\,, \qquad p^{\underline{a}} := \frac{1}{2}(p_1^{\underline{a}} - p_2^{\underline{a}})\,. \tag{6.1}$$

Then using the constraint (5.21) we get from (5.15)

$$H^{(2)} = \xi^{\underline{a}} \cdot \xi^{\underline{a}} \tag{6.2}$$

and

$$p_i = \xi_i + \frac{\lambda}{4\pi}(\xi^{\underline{a}} \cdot \xi^{\underline{a}})\partial_i\,\Phi(\vec{x})\,. \tag{6.3}$$

The Hamiltonian equations take the form

$$\dot{x}^i = \frac{\partial H}{\partial p^i} = 2\left(\xi^{\underline{a}} \cdot \frac{\partial \xi^{\underline{a}}}{\partial p^i}\right), \tag{6.4a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -\,2\left(\xi^{a} \cdot \frac{\partial \xi^{a}}{\partial x^i}\right). \tag{6.4b}$$

Using (6.3) we have

$$\xi^{\underline{a}} \cdot \frac{\partial \xi^{\underline{a}}}{\partial p^i} = \frac{\xi_i}{1 + \frac{\lambda}{2\pi}(\xi^{\underline{a}} \cdot \partial_a \Phi)}, \tag{6.5a}$$

$$\xi^{a} \cdot \frac{\partial \xi^{a}}{\partial x^j} = -\frac{\lambda}{4\pi}\,\frac{(\xi_a\xi^a)\,\xi^i\partial_i\partial_j\Phi}{1 + \frac{\lambda}{2\pi}(\xi^{a} \cdot \partial_a \Phi)}. \tag{6.5b}$$

Taking the time derivative of (6.3) and using (6.4–5) we get

$$\dot{\xi}_i + \frac{\lambda}{2\pi}\partial_i\Phi(x)\xi_j\dot{\xi}_j = 0\,. \tag{6.6}$$

Let us now illustrate for $N = 2$ the procedure leading to the free motion (5.20). Instead of the canonical variables (x_i, p_i) we can use the variables (x_i, ξ_i). Then the Lagrangian obtained from the Hamiltonian (6.2) would have had the form:

$$\begin{aligned} L &= p_l(\xi_i, x_i) \cdot \dot{x}_l - H \\ &= \left(\xi_l + \frac{\lambda}{4\pi}\xi^2\partial_l\Phi(x)\right)\dot{x}_l - \xi^2\,. \end{aligned} \tag{6.7}$$

The variation with respect to ξ_i is given by the expression

$$\xi_i = \frac{\frac{1}{2}\dot{x}_i}{1 - \frac{\lambda}{4\pi}\left(\dot{x}_j \partial_j \Phi\right)}, \tag{6.8}$$

which is equivalent to the Hamiltonian eq. (6.4a) with the insertion of (6.5a). Inserting (6.8) in (6.7) we get

$$L = \frac{1}{4}\,\frac{\dot{x}_l^2}{1 - \frac{\lambda}{4\pi}\left(\dot{x}_l \partial_l \Phi\right)}\,. \tag{6.9}$$

In particular if we observe that

$$\begin{aligned} \det\left(\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j}\right) &= \frac{1}{4}\left(1 - \frac{\lambda}{4\pi}\left(\dot{x}_l \partial_l \Phi\right)\right)^{-4} \\ &= \frac{1}{4}\left(1 + \frac{\lambda}{2\pi}\xi_l \partial_l \Phi\right)^{+4}, \end{aligned} \tag{6.10}$$

we see that if the velocities are expressible in terms of the canonical variables then from (6.6) it follows that

$$\dot{\xi}_l = 0\,. \tag{6.11}$$

Using the Hamiltonian (6.2) we get for the pair of noncanonical variables the Hamilton equations (6.4a) in the form $\dot{x}_l = \{x_l, H\}$ as well as (6.11) given by $\dot{\xi}_l = \{\xi_l, H\}$, provided that we assume the following nonstandard symplectic structure:

$$\left\{x^i, x^j\right\} = \{\xi_i, \xi_j\} = 0, \tag{6.12}$$

$$\left\{x^i, \xi_j\right\} = \delta^i{}_j - \frac{\frac{\lambda}{2\pi}\xi^i \partial_j \Phi}{1 + \frac{\lambda}{2\pi}\left(\xi^a \partial_a \Phi\right)}. \tag{6.13}$$

It is easy to check that the Poisson brackets (6.12-6.13) satisfy the Jacobi identity.

6.2 Two Conserved Angular Momenta

For our system we have two conserved scalar angular momenta (cp. section 4.2):

i) If we define (in $d = 2$ $\quad \vec{a} \wedge \vec{b} = \epsilon_{ij} a^i b^j$ is a scalar)

$$l := \vec{x} \wedge \vec{p} \; = \vec{x} \wedge \vec{\xi} + \frac{\lambda}{4\pi} H \tag{6.14}$$

we find that l is conserved

ii) Second conserved angular momenta $\bar{l}$ is the following

$$\bar{l} := \vec{x} \wedge \vec{\xi} \tag{6.15}$$

because

$$\frac{d}{dt}\bar{l} \;=\; \dot{\vec{x}} \wedge \vec{\xi} \;=\; 0. \tag{6.16}$$

Using the relation

$$\xi^{\underline{a}} \cdot \partial_a \Phi \;=\; -\vec{\xi} \wedge \vec{\nabla} \, \ln r \;=\; \frac{\bar{l}}{r^2}, \tag{6.17}$$

where $r := |\vec{x}|$, we see that (6.8) can be rewritten as:

$$\dot{x}^{\underline{a}} \;=\; \frac{2\xi^{\underline{a}}}{1 + \frac{\lambda \bar{l}}{2\pi r^2}}. \tag{6.18}$$

Note that if $\vec{x}(0)$ is parallel to $\vec{\xi}$ we have $\bar{l} = 0$ and, in consequence, a free motion on a line.

For $\lambda \bar{l} < 0$ it is convenient to introduce the quantity

$$r_0^2 := \frac{|\lambda \bar{l}|}{2\pi} \tag{6.19}$$

We see from (6.18) that the relative two–body problem separates in this case into a motion within the interior region given by

$$r < r_0 \tag{6.20a}$$

and a motion in the exterior region given by

$$r > r_0. \tag{6.20b}$$

6.3 Structure of the classical phase-space M

Let us denote by M the phase space for the canonical variables $(\vec{x}, \vec{p})$. First of all, let us observe that

$$M \neq R^2 \otimes R^2.$$

To show this we start with (6.3) rewritten as

$$\vec{\xi} \;=\; \vec{p} - \frac{\lambda}{4\pi} H \, \vec{\bigtriangledown} \phi. \tag{6.21}$$

Squaring it we get

$$\left(\frac{\lambda}{4\pi} \frac{H}{r} - \left(\frac{2\pi r}{\lambda} + \frac{l}{r} \right) \right)^2 + \vec{p}^2 - \left(\frac{2\pi r}{\lambda} + \frac{l}{r} \right)^2 = 0, \tag{6.22}$$

where we have used the definition (6.14) of l. Thus we must have

$$\vec{p}^2 - \left(\frac{2\pi r}{\lambda} + \frac{l}{r}\right)^2 \leq 0 \tag{6.23}$$

and so (with $l = rp\sin\varphi$)

$$\begin{aligned} r \geq & \quad \tfrac{\lambda}{2\pi}\, p\,(1 - \sin\varphi), \qquad \text{for} \quad \lambda > 0, \\ r \geq & \quad \tfrac{\lambda}{2\pi}\, p\,(1 + \sin\varphi) \qquad \text{for} \quad \lambda < 0, \end{aligned} \tag{6.24}$$

where the equality sign holds for $r = r_0$. Therefore the map $(\vec{x}, \vec{\xi}) \to (\vec{x}, \vec{p})$ with $(\vec{x}, \vec{\xi}) \in R^2 \otimes R^2$ leads to $(\vec{x}, \vec{p}) \in M \neq R^2 \otimes R^2$.

From (6.22) we may look for the Hamiltonian H as a function of the canonical variables $(\vec{x}, \vec{p})$

$$H = \frac{4\pi r}{\lambda}\left[\frac{2\pi r}{\lambda} + \frac{l}{r} \pm \sqrt{\left(\frac{2\pi r}{\lambda} + \frac{l}{r}\right)^2 - \vec{p}^{\,2}}\right] \tag{6.25}$$

with the $-(+)$ sign in front of the second term in (6.25) for $\left(\frac{2\pi r}{\lambda} + \frac{\bar{l}}{r}\right) > 0$ (< 0). Let us note, however, that due to formula (6.14), the RHS of (6.25) depends through l also on the energy H. Writing (6.22) as

$$\left(\frac{\bar{l}}{r} + \frac{2\pi r}{\lambda}\right)^2 + \vec{p}^{\,2} - \left(\frac{2\pi r}{\lambda} + \frac{\bar{l}}{r} + \frac{\lambda}{4\pi r}H\right)^2 = 0 \tag{6.26}$$

we get

$$H = \frac{4\pi r}{\lambda}\left[-\left(\frac{2\pi r}{\lambda} + \frac{\bar{l}}{r}\right) \pm \sqrt{\vec{p}^{\,2} + \left(\frac{2\pi r}{\lambda} + \frac{\bar{l}}{r}\right)^2}\,\right]. \tag{6.27}$$

The formulae (6.25) and (6.27) can be used alternatively in the quantization procedure, depending on the choice of the eigenstates of angular momenta (l in the case of eq. (6.25) and $\bar{l}$ in the case of eq. (6.27)).

6.4 Classical Trajectory

To find the classical trajectories we can make the ansatz

$$\vec{x}(t) \;=\; \alpha(t)\,\vec{\xi} + \beta(t)\,\vec{\xi}_\perp, \tag{6.28}$$

where we have written $\vec{x}$ for $x^{\underline{a}}$ and similarly to $\vec{\xi}$ and defined $\vec{\xi}_\perp$ as a unit vector perpendicular to $\vec{\xi}$. Then

$$\vec{\xi} \wedge \vec{\xi}_\perp \;=\; |\vec{\xi}| \;=\; \xi. \tag{6.29}$$

As ξ is conserved, the conservation of $\bar{l}$ tells us that β is constant and so (6.28) becomes

$$\vec{x}(t) \;=\; \alpha(t)\,\vec{\xi} \,+\, \beta\,\vec{\xi}_\perp \tag{6.30}$$

with a constant β. So the problem has been reduced to finding $\alpha(t)$. To do this we choose a frame in which $\vec{\xi} = \xi\,\vec{e}_x$. Then $\vec{\xi}_\perp = \vec{e}_y$ and we see that

$$\phi(\vec{x}) \;=\; \arctan\left(\frac{\beta}{\alpha(t)\xi}\right). \tag{6.31}$$

Next, as in the previous section, we introduce $\vec{y}$

$$\vec{y} \;=\; \vec{x} \,+\, \frac{\lambda}{2\pi}\,\vec{\xi}\,\phi(\vec{x}), \tag{6.32}$$

which is canonically conjugate to $\vec{\xi}$, and find that using the last two formulae we see that $\alpha(t)$ satisfies

$$2\,\vec{\xi}t \,+\, \vec{y}(0) \;=\; \alpha(t)\vec{\xi} \,+\, \beta\vec{\xi}_\perp \,+\, \frac{\lambda}{2\pi}\,\vec{\xi}\,\arctan\left(\frac{\beta}{\alpha(t)\xi}\right), \tag{6.33}$$

i.e. we arive at a fixed point equation for $\alpha(t)$

$$\alpha(t) \;=\; -\frac{\lambda}{2\pi}\,\arctan\left(\frac{\beta}{\alpha(t)\xi}\right) \,+\, 2t \,+\, c, \tag{6.34}$$

with c determined by the initial conditions.

To discuss the conditions on the existence of $\alpha(t)$ which solve (6.34) we rescale it by introducing

$$\tau = 2\,\frac{t\xi}{\beta}, \quad k = -\frac{\lambda\xi}{2\pi\beta}, \quad \text{and} \quad g(\tau) = 2\,\frac{\alpha(t)\xi}{\beta}. \tag{6.35}$$

Then after a time translation $t + c \rightarrow t$ our equation (6.34) becomes

$$g \;=\; f(g) \,+\, \tau, \tag{6.36}$$

where

$$f(g) \;=\; k\arctan\left(\frac{1}{g}\right). \tag{6.37}$$

To discuss the solvability of (6.36) it is convenient to redefine f to be given by

$$f(g) \;=\; -k\arctan g \tag{6.38}$$

and perform a further time translation so that the equation for g is still of the form (6.36). Then putting

$$g \;=\; \nu \,+\, \tau \tag{6.39}$$

we find that (6.36) becomes

$$\nu \;=\; -k\int_0^{\nu+\tau}\,dx\,\frac{1}{1+x^2}\;=:\;g_\tau(\nu). \tag{6.40}$$

Note that this implies that

$$\nu(-\infty)\;=\;k\,\frac{\pi}{2} \tag{6.41}$$

and

$$\nu(\infty)\;=\;-k\,\frac{\pi}{2} \tag{6.42}$$

independently of the value of k.

Note further that

$$\frac{\partial g_\tau}{\partial\nu}(\nu)\;=\;-k\,\frac{1}{1+(\nu+\tau)^2} \tag{6.43}$$

Let us consider first the case of $k>-1$.

We rewrite (6.40) as

$$G_\tau(\nu) :=\;\nu\;-\;g_\tau\;=\;0. \tag{6.44}$$

Then from (6.40) we have

$$G_\tau(\pm\infty)\;=\;\pm\infty. \tag{6.45}$$

Differentiating (6.26) with respect to τ we find

$$\dot\nu(\tau)\;=\;-\frac{k}{(\nu(\tau)+\tau)^2+1+k} \tag{6.46}$$

From (6.43) we conclude that $G_\tau(\nu)$ is a monotonically increasing function of ν. Given (6.45) and (6.46) this implies that there is a unique and smooth solution $\nu(\tau)$ of (6.44) for each $\tau\in R^1$.

For $k<-1$ we have a problem as ν develops a discontinuity. This can be seen by plotting the various terms in (6.36). Thus we see that for $k>-1$ we have a well defined $g(\tau)$ while for $k<-1$ the function $g(\tau)$ "jumps" at some value of τ showing that the x space is not the physical configuration space in this case.

Let us discuss this in more detail.

Eq. (6.46) shows the smoothness of $\nu(\tau)$ for all values of $k>-1$. For $k<-1$, however, $\nu(\tau)$ has an infinite slope at $\tau=\pm\tau_0$, where

$$\tau_0 := -\sqrt{|k|-1}\;+\;|k|\int_0^{\sqrt{|k|-1}}dx\,\frac{1}{1+x^2}. \tag{6.47}$$

But due to the symmetry $\nu(-\tau)=-\nu(\tau)$ it is sufficient to consider the case of $\tau=\tau_0$. To do this we note that if we start at $\tau=-\infty$ we have a smooth function $\nu(\tau)$ for $\tau<\tau_0$. Close to $\tau=\tau_0$ we have

$$\lim_{\epsilon\to+0}\dot\nu(\tau_0-\epsilon)\;=\;+\infty \tag{6.48}$$

with

$$\nu(\tau_0 - \epsilon) \;=\; -\tau_0 \;-\; \sqrt{|k| - 1} \;+\; \epsilon \tag{6.49}$$

thus showing that $\nu(\tau)$ jumps at $\tau = \tau_0$ to

$$\nu(\tau_0 + \epsilon) \;>\; \nu(\tau_0 - \epsilon). \tag{6.50}$$

However, the states $\nu(\tau_0 \pm \epsilon)$ are physically equivalent; *i.e.* they have to be identified in the configuration space C_{phys}. (6.33) shows that C_{phys} is determined by y-space defined by (6.32). To see this we recall from section 5.3 that the map $\vec{x} \to \vec{y}$ is part of a canonical transformation iff

$$\det\left(\frac{\partial^2 F}{\partial x_i \partial \xi_j}\right) \;\neq\; 0 \tag{6.51}$$

which for $N = 2$ is equivalent to

$$1 \;+\; \frac{\lambda}{2\pi}\, \xi^i\, \partial_i \phi \;\neq\; 0 \tag{6.52}$$

and so

$$(\nu \;+\; \tau)^2 \;+\; 1 \;+\; k \;\neq\; 0. \tag{6.53}$$

But for the whole trajectory $\nu(\tau)$ we have (with the identification $\nu(\tau_0 - \epsilon) \sim \nu(\tau_0 + \epsilon)$)

$$(\nu \;+\; \tau)^2 \;+\; 1 \;+\; k \;>\; 0, \tag{6.54}$$

which, due to (6.17) is equivalent to

$$r^2 > r_0^2 \qquad \text{for} \quad \lambda \bar{l} < 0. \tag{6.55}$$

Thus we have a one-to-one correspondence between the y-space and C_{phys} for $r > r_0$.

Within the interior of the region $r < r_0$ we obtain from (6.40) a solution valid only for a finite time interval. Such a situation is well known in General Relativity (cp. [38]) and the literature cited there).

From the physical point of view the case $k < -1$ is the most interesting as then the configuration space consists of two non-communicationg regions; the exterior ($r > r_0$) and the interior ($r < r_0$). Thus, in the next section we will concentrate on the quantization of this case.

7 Quantization of the Two–Body Problem

7.1 Nonstandard Schrödinger Equation

The relation (6.3), after the substitution of (6.2), takes the form ($H \equiv H^{(2)}$):

$$\vec{\xi} = \vec{p} - \frac{\lambda}{4\pi} H \vec{\bigtriangledown} \Phi. \tag{7.1}$$

Squaring it and using again (6.2) we obtain

$$H = \vec{p}^2 - \frac{l^2}{r^2} + \frac{\bar{l}^{\,2}}{r^2}, \tag{7.2a}$$

where we have used the definition (6.14) of $\bar{l}$.

Let us note that

$$\bar{l} = l - \frac{\lambda}{4\pi} H \tag{7.2b}$$

i.e. (7.2a) gives us a **quadratic** equation for H (cp. section 6.3).

We quantize the problem by considering a Schrödinger-like equation

$$i\hbar \frac{\partial \psi(\vec{x},t)}{\partial t} = \hat{H}\, \psi(\vec{x},t) = \left[\hat{\vec{p}}^{\,2} - \frac{l^2}{r^2} + \frac{1}{r^2} \bar{l}^{\,2} \right] \Psi(\vec{x},t) \tag{7.3}$$

in which the operators $\hat{H}$ and $\hat{\vec{p}}$ are defined by the usual quantization rules

$$\hat{H} = i\hbar \frac{\partial}{\partial t}, \qquad \hat{p}_l = \frac{\hbar}{i} \partial_l\, . \tag{7.4}$$

We see that the equation (7.3) describes a nonstandard form of a time dependent Schrödinger equation, with its right hand side containing both the first and second time derivatives (entering through $\bar{l}$).

For the stationary case, *i.e.* when $\Psi(\vec{x},t) = \Psi_E(\vec{x}) e^{\frac{iEt}{\hbar}}$ we can use the angular-momentum basis and put

$$\Psi_{E,m} = f_{E,m}(r)\, e^{im\varphi} \tag{7.5}$$

where m is an integer, and find that $f_{E,m}$ satisfies a nonstandard time independent Schrödinger equation

$$\left[-\hbar^2 \left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{\bar{m}^2}{r^2} \right) - E \right] f_{E,m}(r) = 0, \tag{7.6}$$

where in consistency with (7.2b) we have defined

$$\hbar \bar{m} := \hbar m - \frac{\lambda}{4\pi} E \tag{7.7}$$

i.e. $\hbar\bar{m}$ is an eigenvalue of $\bar{l}$.

A characteristic feature of the Schrödinger equation (7.6) is the appearance of the noncanonical angular momentum $\hbar\bar{m}$ with $\bar{m}$ not being an integer. *i.e.* our two-body system carries a fractional orbital angular momentum (see the discussion in [27] or [39]). It has been shown in section 4.2 that $J = \bar{l}$ is equal to the total angular

momentum of the particle + field system with a nonvanishing angular momentum of the fields. We note that a form of (7.7) shows a great similarity bteween our two-particle state and the gravitational anyon of Cho et al. [40] as an energy-spin composite.

In the following we discuss only the most interesting case of $\lambda\bar{l} < 0$.

Now the appropriate boundary conditions correspond to the requirement that $f_{E,m}(r)$ is nonzero in either the interior region ($r < r_0$) or in the exterior region ($r > r_0$). Thus our boundary condition is

$$f_{E,m}(r_0) = 0 \tag{7.8}$$

The general solution of (7.6) is given by

$$f_{E,m}(r) = Z_{\overline{m}}\left(\frac{\sqrt{E}}{\hbar}\,r\right), \tag{7.9}$$

where $Z_{\overline{m}}$ is an appropriate Bessel function of order $\overline{m}$ (or a superposition of such functions).

7.2 Interior Solutions ($r < r_0$)

The only Bessel functions not blowing up as $r \to 0$, *i.e.* giving a finite probability $f^2\,r\,dr$ as $r \to 0$, and possessing positive zeros r_0 are those of the first kind of order $|\bar{m}|$ with $E \geq 0$.

Then the possible eigenvalues $E_n(m)$ are determined by

$$J_{\bar{m}}\left[\frac{\sqrt{E}}{\hbar}\left(\frac{\hbar|\lambda\bar{m}|}{2\pi}\right)^{\frac{1}{2}}\right] \;=\; 0 \tag{7.10}$$

with $\bar{m}$ given by (7.7).

Let us look in more detail at the case $\bar{m} > 0$, $\lambda < 0$. To simplify (7.10) we define

$$\epsilon \;=\; \frac{|\lambda|E}{2\pi\hbar}. \tag{7.11}$$

Thus (7.10) takes the form

$$J_{\bar{m}}(\bar{m}^{\frac{1}{2}}\epsilon^{\frac{1}{2}}) \;=\; 0. \tag{7.12}$$

As $J_{\bar{m}}$, for fixed $\bar{m} > 0$, has an infinite number of positive zeros, which we denote by $y_n(\bar{m})$, $n = 1, 2..$ we see that due to (7.7), the eigenvalues $\epsilon_n(m)$ we are looking for are the positive fixed points of the equation

$$\epsilon \;=\; f_n(m + \frac{1}{2}\epsilon), \tag{7.13}$$

where we have defined

$$f_n(\bar{m}) \;=\; \frac{1}{\bar{m}} y_n^2(\bar{m}). \tag{7.14}$$

The existence of positive fixed points ϵ of (7.13) may be shown by using appropriate bounds for f_n or y_n. For example for $n = 1$ and $\bar{m} > 0$ we have [41]

$$\bar{m}(\bar{m}+2) \;<\; \bar{m} f_1(\bar{m}) \;<\; \frac{4}{3}(\bar{m}+1)(\bar{m}+5) \tag{7.15}$$

and hence we have, with $\bar{m} = m + \frac{1}{2}\epsilon$ and for $m \in N$,

$$2(m+2) \;<\; \epsilon_1(m) \;<\; m + 12 \;+\; \Big[(m+12)^2 + 8(m+1)(m+5)\Big]^{\frac{1}{2}}, \tag{7.16}$$

i.e. we have proved the existence of $\epsilon_1(m) > 0$ for $m = 1, 2..$ and give crude bounds for it. A better estimate for large m can be obtained by using the asymptotic formula [41]

$$y_1(\bar{m}) \;=\; \bar{m} \;+\; 1.855757\bar{m}^{\frac{1}{3}} \;+\; O(\bar{m}^{-\frac{1}{3}})$$

valid for large $\bar{m}$ leading to

$$\epsilon_1(m) \;=\; 2m \;+\; 9.35243 m^{\frac{1}{3}} \;+\; O(m^{-\frac{1}{3}}) \tag{7.17}$$

valid for large positive m.

To get more insight the equation (7.13) has to be solved numerically. To do this we have to determine the behaviour of zeros of the Bessel function J_k as a function of k. Then having determined this dependence we can find the values of energy by the secant method.

Luckily there are many computer programs to determine zeros of Bessel functions and in our work we have used the Maple program to perform this task. As the dependence of each zero is almost linear the numerical procedure of solving (7.13) converges rapidly.

In fig.1 we present the values of energy for $\epsilon \leq 300$ as a function of m. The plot looks like several curves; the lowest values correspond to first zeros (*ie* $n = 1$), the next ones to second zeros *ie* $n = 2$ *etc.* The points lie so close that the figure may appear as a set of lines while, in reality, we have here sets of points. The points appear to be (almost) equally spaced on each "curve" - this is due to the approximate linearity of the positions of zeros of Bessel functions as a function of $\bar{m}$. To check our values of energies we have also solved (7.13) differently; we approximated the positions of zeros of the Bessel functions by a linear function and solved the resultant equations for ϵ. The obtained results were very similar to what is shown in fig. 1 thus giving us confidence in our results.

Our results show that, for each value of m, there is a whole tower of values of ϵ corresponding to different zeros of the Bessel functions. The values of ϵ increase, approximately linearly, as we take higher zeros (*ie* y_n for larger n). The dependence

on m is only slightly more complicated; for each order of the zero there is a value of m for which the energy is minimal and as we move away from this value the energy grows, approximately, linearly. As n increases the minimal values of m increase, again, approximately linearly.

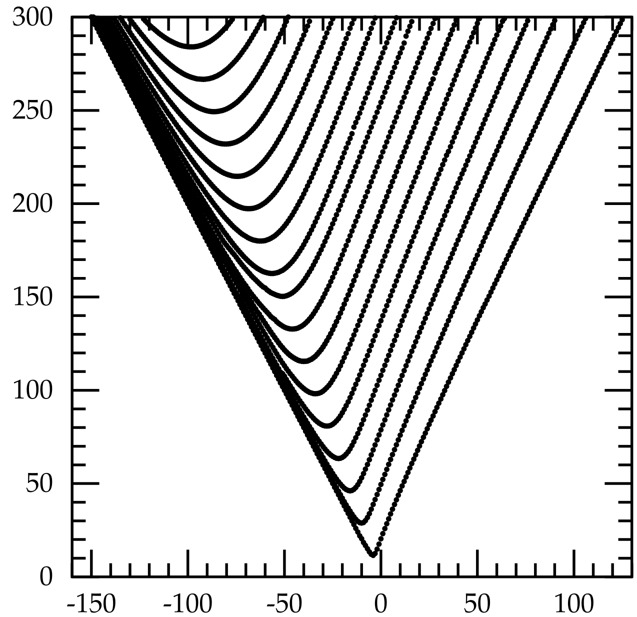

Figure 1: Energy as a function of m

Note that our numerical values of ϵ are consistent with the asymptotic results mentioned before. Note also that for $\bar{m} < 0$ and $\lambda > 0$ the corresponding energy levels are obtained by changing the sign of m.

Summarising, we see that in the interior region $r < r_0$, where classical solutions are only possible for a finite time interval, we find quantum solutions which correspond to discrete bound states determined by the boundary condition at $r = r_0$. Thus we see that this boundary condition defines a geometric "bag" for the quantum state.

7.3 Exterior Solutions $(r > r_0)$

First of all, let us note that there are no bound states solutions of (7.6) for $r > r_0$ as the only square integrable Bessel functions in $[r_0, \infty)$ are the modified Bessel functions of the third kind, which have neither positive nor pure imaginary zeros (see e.g. [41]).

Scattering solutions are given by a superposition of Bessel functions of the first and second kind

$$f_{E,m}(r) \,=\, A_m(E)\, J_{\bar m}\left(\frac{\sqrt{E}}{\hbar} r\right) \,+\, B_m(E)\, Y_{\bar m}\left(\frac{\sqrt{E}}{\hbar} r\right) \tag{7.18}$$

with the ratio $\frac{A_m}{B_m}$ determined by the boundary condition (7.8). These solutions describe scattering on an obstruction of radius r_0, which is dynamically determined.

8 Two Charged Gravitationally Interacting Particles in a Magnetic Field

In this section we consider the motion considered in Sect. 6 and 7 of two particles, of equal electric charge e, under the influence of an additional static uniform magnetic field B perpendicular to the plane of motion. We assume, for simplicity, that $eB > 0$.

8.1 Classical Dynamics

To have the Lagrangian L, describing the relative motion of two charged particles in an additional constant magnetic field B we add to the Lagrangian considered in Section 6 the term

$$L_{magn} \,=\, \frac{1}{4}\, e\, B\, \epsilon_{ij}\, x_i\, \dot{x}_j. \tag{8.1}$$

We obtain

$$L \,=\, (\xi_i \,+\, \frac{\lambda}{4\pi}\, \xi^2\, \partial_i\, \phi)\dot{x}_i \,-\, \xi^2 \,+\, \frac{1}{4}\, e\, B\, \epsilon_{ij}\, x_i\, \dot{x}_j \tag{8.2}$$

and therefore, in accordance with (6.2) we have

$$H \,=\, \vec{\xi}^{\,2}\,. \tag{8.3}$$

The equations of motion for x_i, derived from (8.2), are the same as before, *ie*

$$\dot{x}_i \,=\, \frac{2\xi_i}{1 + \frac{\lambda}{2\pi}\, \xi_l\, \partial_l\, \phi}, \tag{8.4}$$

where the covariant velocities ξ_i for $B \neq 0$ are not conserved. However, as ξ_i satisfy

$$\dot{\xi}_i \,=\, \frac{eB\epsilon_{ij}\xi_j}{1 + \frac{\lambda}{2\pi}\xi_l\partial_l\phi} \tag{8.5}$$

we see (from (8.4-5)) that the conserved quantities are now

$$\tilde{\xi}_i := \xi_i - \frac{eB}{2}\,\epsilon_{ij}\,x_j. \tag{8.6}$$

From the equations of motion (8.4-5) and the Hamiltonian (8.3) we derive the following, nonstandard, symplectic structure for the noncanonical set of variables $(\vec{x}, \vec{\xi})$ or $(\vec{x}, \vec{\tilde{\xi}})$, respectively:

$$\{x_i,\, x_j\} \,=\, 0 \tag{8.7a}$$

$$\{x_i,\, \xi_j\} \,=\, \{x_i,\, \tilde{\xi}_j\} \,=\, \delta_{ij} \,-\, \frac{\frac{\lambda}{2\pi}\,\xi_i \partial_j \phi}{1 + \frac{\lambda}{2\pi}\xi_l \partial_l \phi} \tag{8.7b}$$

$$\{\xi_i,\, \xi_j\} \,=\, \frac{\frac{1}{2} eB\epsilon_{ij}}{1 + \frac{\lambda}{2\pi}\xi_l \partial_l \phi} \tag{8.7c}$$

but

$$\{\tilde{\xi}_i,\, \tilde{\xi}_j\} \,=\, -\frac{1}{2} eB\epsilon_{ij}. \tag{8.7d}$$

It is easy to check that the Poisson brackets (8.7a-d) satisfy the Jacobi identities.

With the canonical momentum p_i derived from (8.2)

$$p_i \,=\, \xi_i \,+\, \frac{\lambda}{4\pi}\,H\,\partial_i \phi \,-\, \frac{eB}{4}\,\epsilon_{ij}\,x_j \tag{8.8}$$

we find that the relation between the two conserved angular momenta $l, \bar{l}$ is not modified

$$l := \vec{x} \wedge \vec{p} \,=\, \bar{l} \,+\, \frac{\lambda}{4\pi} H; \tag{8.9}$$

however, $\bar{l}$ is given by

$$\bar{l} := \vec{x} \wedge \vec{\xi} + \frac{1}{4}\, e\, B\, r^2\, . \tag{8.10}$$

We note that due to (8.9-10) the form of the denominator in the equation of motion (8.4-5) is different when compared with the $B = 0$ case:

$$1 \,+\, \frac{\lambda}{2\pi}\,\xi_l \partial_l \phi \,=\, \left(1 - \frac{\lambda}{8\pi} eB\right) \,+\, \frac{\lambda \bar{l}}{2\pi r^2}. \tag{8.11}$$

Let us consider just the nonconfined motion, *ie* (8.11) should be nonvanishing for all $0 < r < \infty$. When $B = 0$ this would correspond to the case of $\lambda \bar{l} > 0$ - *ie* "the less interesting case" of the previous section.

Now, however, we have to distinguish two subcases:

Either

$$\text{A}: \qquad 1 - \frac{\lambda}{8\pi}\, eB \,>\, 0, \tag{8.12a}$$

which allows both signs of λ,

$$\text{and} \quad \lambda\bar{l} \,>\, 0 \tag{8.12b}$$

or

$$\text{B}: \qquad 1 - \frac{\lambda}{8\pi}\, eB \,<\, 0, \tag{8.13a}$$

which allows only $\lambda > 0$

$$\text{and} \quad \bar{l} \,<\, 0. \tag{8.13b}$$

Note that when $B \neq 0$ we can have a nonconfined motion also for $\lambda\bar{l} < 0$.

In analogy to the free field case we introduce for

$$\frac{\lambda\bar{l}}{1 - \frac{\lambda}{8\pi} eB} \,<\, 0 \tag{8.14}$$

the boundary r_0 between the interior ($r < r_0$) and the exterior ($r > r_0$) region defined by

$$r_0^2 \,=\, \frac{1}{2\pi}\left|\frac{\lambda\bar{l}}{1 - \frac{\lambda}{8\pi} eB}\right|. \tag{8.15}$$

When we consider the confined motion we consider the motion in the interior region ($r < r_0$). In this case we have to consider two further subclasses:

Either

$$\text{C}: \qquad 1 - \frac{\lambda}{8\pi}\, eB \,>\, 0, \tag{8.16a}$$

which allows both signs of λ,

$$\text{and} \quad \lambda\bar{l} \,<\, 0 \tag{8.16b}$$

or

$$\text{D}: \qquad 1 - \frac{\lambda}{8\pi}\, eB \,<\, 0, \tag{8.17a}$$

which allows only $\lambda > 0$

$$\text{and} \quad \bar{l} \,>\, 0. \tag{8.17b}$$

Note that when $B \neq 0$ we can have a confined motion also for $\lambda\bar{l} > 0$.

Comparing all four subcases we conclude that we obtain confinement $\leftrightarrow$ nonconfinement transitions by tuning the strength of the magnetic field B. To be more specific let us consider *eg* $\lambda > 0$ and $\bar{l} > 0$. Then we obtain, for eB increasing from low values towards $\frac{8\pi}{\lambda}$, a transition from case A to case D *ie* a nonconfinement $\rightarrow$ confinement transition. This transition is continuous because we have $r_0 = \infty$ at the transition point.

In order to quantize our system we have to generalize for $B \neq 0$ the relation (7.2a). We start with (8.8) rewritten as

$$\xi_i \,=\, p_i \,-\, \frac{\lambda}{4\pi}\, H\, \partial_i\, \phi \,+\, \frac{eB}{4}\, \epsilon_{ij}\, x_j, \tag{8.18}$$

square it and then using (8.3) and (8.9-10) we find for the Hamiltonian H

$$H = p^2 - \frac{l^2}{r^2} + \frac{\bar{l}^2}{r^2} + \frac{1}{16}(eB)^2 r^2 - \frac{1}{2} eB\bar{l}. \tag{8.19}$$

8.2 Quantization

Following the method presented in Section 7.1 we obtain from (8.19) the following nonstandard Schrödinger equation for the radial wave function $f_{E,m}$

$$\left[-\hbar^2 \left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\bar{m}^2}{r^2}\right) + \left(\frac{eB}{4}\right)^2 r^2 - E - \frac{\hbar}{2}eB\bar{m}\right] f_{E,m}(r) = 0, \tag{8.20}$$

which generalizes (7.6). Like in the previous case, the noninteger eigenvalue $\bar{m}$ is related to the integer m and energy E by the eq. (7.7).

8.2.1 Nonconfined motion

The eigenvalue problem (8.20) is identical to the two anyon problem in a constant B field with the statistics parameter being proportional to the energy value E. So, for the energy levels $E_{n,m}$ we obtain ([42-44])

$$E_{n,m} = \hbar\, e\, B\, (n + \frac{1}{2}|\bar{m}| + \frac{1}{2}) - \frac{\hbar}{2}eB\bar{m}, \tag{8.21}$$

with $n = 0, 1, 2, ..$ Thus, according to the two cases A and B defined above and for different signs of λ we have to distinguish between three cases:

Case A with $\lambda > 0$

Due to (8.12b) and (7.7) we have

$$\bar{m} = m - \frac{\lambda}{4\pi\hbar} E_{n,m} > 0. \tag{8.22}$$

Therefore we get from (8.21)

$$E_{n,m} = \hbar eB(n + \frac{1}{2}) \tag{8.23}$$

corresponding to the case II in [43].

Combining (8.22) and (8.23) gives us

$$m > \frac{\lambda eB}{4\pi}\left(n + \frac{1}{2}\right), \tag{8.24}$$

where, due to (8.12a), the product of the coupling strength λ and of the field strength B is bounded

$$\lambda eB < 8\pi. \tag{8.25}$$

Case B with $\lambda > 0$

Due to (8.13b) and (7.7) we have

$$\bar{m} \,=\, m \,-\, \frac{\lambda}{4\pi\hbar}\, E_{n,m} \,<\, 0. \tag{8.26}$$

Thus, from (8.21) we obtain

$$E_{n,m} \,=\, \hbar e B \left(n \,+\, |\bar{m}| \,+\, \frac{1}{2} \right), \tag{8.27}$$

which corresponds to the case I in [43].

Inserting (8.26) into (8.27) we obtain

$$E_{n,m} \,=\, \frac{\hbar e B}{1 - \frac{\lambda e B}{4\pi}} \left(n - m + \frac{1}{2} \right). \tag{8.28}$$

Note that in order to satisfy (8.13a) the product of the coupling strength λ and of the field strength B must be above the mininum value

$$\lambda e B \,>\, 8\pi. \tag{8.29}$$

Therefore we obtain from (8.26) and (8.28)

$$m \,>\, \frac{\lambda e B}{4\pi} \left(n \,+\, \frac{1}{2} \right). \tag{8.30}$$

Thus we see that $E_{n,m}$ is bounded from below by

$$E_{n,m} \,>\, \hbar e B (n + \frac{1}{2}). \tag{8.31}$$

Case A with $\lambda < 0$

The condition (8.12a) is now fulfilled automatically. From (8.12b) we obtain (8.26) and therefore we have the result identical to (8.28).

Now we obtain

$$m \,<\, -\frac{|\lambda| e B}{4\pi} (n + \frac{1}{2}) \tag{8.32}$$

and $E_{n,m}$ is bounded from below again by

$$E_{n,m} \,>\, \hbar e B (n + \frac{1}{2}). \tag{8.33}$$

8.2.2 Confined motion

Now we have to consider a nonstandard Schrödinger equation (8.20) in the interior region ($r < r_0$) with the boundary condition as in the previous case

$$f_{E,m}(r_0) \;=\; 0. \tag{8.34}$$

The regular solution of (8.20) is given, up to a normalisation factor, by [45]

$$f_{E,m}(r) \;=\; r^{|\bar{m}|}\, e^{-\frac{\beta r^2}{2}} \phi\left(\frac{|\bar{m}| - \bar{m} + 1}{2} - \gamma, 1 + |\bar{m}|; \beta r^2\right) \tag{8.35}$$

with

$$\beta \;=\; \frac{eB}{4\hbar} \tag{8.36a}$$

and

$$\gamma \;=\; \frac{E}{\hbar eB} \tag{8.36b}$$

and where $\phi(a, b; z)$ denotes the confluent hypergeometric function.

From (8.15), (8.34) and (8.35) we conclude that our energy levels $E_{n,m}$ are determined by the roots of the equation, which only be solved numerically,

$$\phi\left(\frac{|\bar{m}| - \bar{m} + 1}{2} - \frac{E}{\hbar eB}, 1 + |\bar{m}|; \frac{eB}{8\pi}\left|\frac{\lambda \bar{m}}{1 - \frac{\lambda}{8\pi} eB}\right|\right) \;=\; 0, \tag{8.37}$$

where $\bar{m}$ is a function of E as given by (7.7).

The zero B field limit

From the well known relation [45]

$$\lim_{a\to\infty} \phi(a, b; -\frac{x}{a}) \;=\; \Gamma(b) x^{\frac{1}{2}(1-b)}\, J_{b-1}(2\sqrt{x}) \tag{8.38}$$

we obtain, from (8.35) and (8.37), in the vanishing B limit, the considered in section 7, respectively, the wave function and the eigenvalue condition.

The high B field limit

Without going into the numerics we can conclude from (8.37) that for $eB \to \infty$ the energy levels increase with the increase of eB, at least, linearly. We prove this statement by assuming the contrary. Then one has at the l.h.s. of (8.37), with $\bar{m} > 0$, due to (8.16-17), $\phi(\frac{1}{2}, 1 + \bar{m}; \bar{m})$ for which we have the inequality

$$\phi(\frac{1}{2}, 1 + \bar{m}; \bar{m}) \;\geq\; 1 \quad \text{for} \quad \bar{m} > 0 \tag{8.39}$$

in contradistinction to (8.37).

Confinement → Nonconfinement transitions

According to (8.15) confinement → nonconfinement transitions occur if $r_0 \to \infty$, *ie* if

$$1 - \frac{\lambda}{8\pi} eB \to 0. \tag{8.40}$$

But from the asymptotic behaviour of the confluent hypergeometric function [45]

$$\phi(a, b; x) \to_{x\to\infty} \frac{\Gamma(b)}{\Gamma(a)} e^x \, x^{a-b} \tag{8.41}$$

we conclude that the r.h.s. vanishes only if $a = 0, -1, -2, ...,$ *ie* for (8.37), if the energy levels are given by (8.21) as required.

9 Conclusions and Outlook

The aim of this paper has been to consider, for two–dimensional nonrelativistic particles, a new interaction scheme, generated by the coupling to the topological torsion Lagrangian. We have shown that only in $D = 2 + 1$ one can write the torsion Lagrangian in the form of a bilinear Chern–Simons term, described by the translational gauge fields (threebeins) multiplied by their Abelian field strength (torsion fields). By a suitable choice of the reparametrization gauge, we have then determined the solutions of the two–body dynamics with fractional angular momentum eigenvalues corresponding to trajectories confined to finite regions of $d = 2$ space. The quantization problem was described by a new type of Schrödinger equation, with a second order time derivative, which in the stationary case gave us a nonlinear energy eigenvalue problem, describing infinite sequence of bound states.

The eigenvalues of energy were determined numerically and we showed that the spectrum was descrete and characterised by two integers; one of them corresponding to the rotational integer quantum number m and the other, another integer - n, described which zero of the Bessel function the state corresponded to. The dependence on both of these integer-value parameters was approximately linear - as we discussed this in section 7.

We have also noted that due to the topological form of the free gravitational field action the case of two dimensions is exceptional. In $d = 3$ one can also look for a modification of the standard gravitational interactions (e.g. of the Newton potential in the lowest order static approximation) by adding to the Hilbert–Einstein action appropriate bilinear torsion terms (see e.g. [7]). However, the confinement due to the presence of torsion in $D = 3 + 1$ gravity leads in the three–dimensional space to an additional potential term of a harmonic form (see e.g. [8]). Such a mechanism of confinement is different from our proposal, where the two-particle Lagrangian becomes singular at the boundary r_0 of a dynamically determined compact region of space. Equivalently, confinement is obtained from the singularity of the nonstandard symplectic structure within the Hamiltonian formalism.

Our considerations have thus shown us [2,3,13] that the geometric bag formation[13] based on our dynamical assumptions are possible in $d = 1$ and $d = 2$ spaces, but it is not clear how to obtain analogous solutions, to be generated by nonstandard gravitational interactions, in the $d = 3$ case, *i.e.* in the standard physical space–time. It is worth pointing out, however, that our geometric bag solutions carry some resemblance to the fields describing black holes (see e.g. the dynamical division of space into two disconnected domains – interior and exterior). At present we can only hope that all such consequences for $d > 2$ will be further clarified in the future development of our approach.

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